Assignment 11.

This homework is due *Thursday*, Nov 19.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 11.

1. Metric spaces. Quick reminder

Metric space is a pair (X, ρ), where X is a nonempty set and ρ is a function $\rho : X \times X \to \mathbb{R}$, called metric, such that $\forall x, y, z \in X$

(1) $\rho(x, y) \geq 0$,
(2) $\rho(x, y) = 0$ if and only if $x = y$,
(3) $\rho(x, y) = \rho(y, x)$,
(4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Normed linear space is a pair $(V, \|\cdot\|)$, where V is a linear space and $\|\cdot\|$ is a function $\|\cdot\| : V \to \mathbb{R}$, called norm, such that $\forall u, v \in V$ and $\forall \alpha \in \mathbb{R}$,

(1) $\|u\| \geq 0$,
(2) $\|u\| = 0$ if and only if $u = 0$,
(3) $\|u + v\| \leq \|u\| + \|v\|$,
(4) $\|\alpha u\| = |\alpha| \|u\|$.

Every norm induces a metric via $\rho(u, v) = \|u - v\|$.

2. Exercises

(1) (9.1.4+)
[This problem already appeared in HW10, skip it.]
(a) Let $X = C[a, b]$. Show that $\|f\|_1 = \int_{[a,b]} |f|$ is a norm.
(b) Show that the norm above is not equivalent to $\|f\|_\infty$. That is, show that there are no constants $c_1, c_2 > 0$ such that $\forall f \in C[a, b]$, $c_1\|f\|_1 \leq \|f\|_\infty \leq c_2\|f\|_1$. Reminder: $\|f\|_\infty = \max\limits_{x\in[a,b]}\{|f(x)|\}$.

(2) (9.2.20–22) For a subset E of a metric space X, a point $x \in X$ is called
- an interior point of E if there is $r > 0$ s.t. $B(x, r) \subseteq E$; the collection of interior points of E is called the interior of E and denoted $\operatorname{int} E$;
- an exterior point of E if there is $r > 0$ s.t. $B(x, r) \subseteq X \setminus E$; the collection of exterior points of E is called the exterior of E and denoted $\operatorname{ext} E$;
- a boundary point of E if for all $r > 0$, $B(x, r) \cap E \neq \emptyset$ and $B(x, r) \cap (X \setminus E) \neq \emptyset$; the collection of boundary points of E is called the boundary of E and denoted $\operatorname{bd} E$ or ∂E.

(a) Prove that $\operatorname{int} E$ is always open and that E is open iff $E = \operatorname{int} E$.
(b) Prove that $\operatorname{ext} E$ is always open and that E is closed iff $X \setminus E = \operatorname{ext} E$.
(c) Prove that $\operatorname{bd} E$ is always closed; that E is open iff $E \cap \operatorname{bd} E = \emptyset$; and that that E is closed iff $\operatorname{bd} E \subseteq E$.

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(3) Let ρ and σ be two equivalent metrics on X.
   (a) Prove that a sequence $\{x_n\}$ converges to x in (X, ρ) if and only if it converges to x in (X, σ).
   (b) Prove that a subset $E \subseteq X$ is open in (X, ρ) if and only if it is open in (X, σ).
   (*Hint:* Actually, (a)$\Leftrightarrow$(b), but proving that is about as much effort as proving them separately.)

(4) (a) (9.3.32i) For a nonempty subset E of a metric space (X, ρ) and a point $x \in X$, define the distance from x to E, $\operatorname{dist}(x, E)$ as follows:
$$\operatorname{dist}(x, E) = \inf\{\rho(x, y) \mid y \in E\}.$$
   Show that the distance function $f : X \to \mathbb{R}$ defined by $f(x) = \operatorname{dist}(x, E)$, for $x \in X$, is continuous.
   (b) (9.3.32ii) Show that $\{x \in X \mid \operatorname{dist}(x, E) = 0\} = \overline{E}$.
   (c) (9.3.34) Show that a subset E of a metric space X is closed if and only if there is a continuous function $f : X \to \mathbb{R}$ for which $E = f^{-1}(0)$.
   (d) (9.3.33) Show that a subset E of a metric space X is open if and only if there is continuous function $f : X \to \mathbb{R}$ for which $E = \{x \in X \mid f(x) > 0\}$.

(5) (9.4.38) In a metric space X, show that a Cauchy sequence converges if and only if it has a convergent subsequence.

(6) ($\sim$9.4.39) Let $0 < \alpha < 1$. Suppose that $\{x_n\}$ is a sequence in a complete metric space (X, ρ) and for each n, $\rho(x_n, x_{n+1}) \leq \alpha^n$. Show that $\{x_n\}$ converges. Does $\{x_n\}$ necessarily converge if we only require that for each n, $\rho(x_n, x_{n+1}) \leq 1/n$?

(7) (9.4.47) Let $\mathcal{D}$ be the subspace of $C[0, 1]$ (with uniform metric) consisting of the continuous functions $[0, 1] \to \mathbb{R}$ that are differentiable on $(0, 1)$. Is $\mathcal{D}$ complete?

3. Extra Problem

(8) Suppose X is a nonempty set and ρ, σ are two metrics on X. Suppose that a sequence $\{x_n\}$ in X converges to x in (X, ρ) if and only if it converges to x in (X, σ). Are ρ and σ are necessarily equivalent? (In other words, is converse to Problem 3a true?)